GROTHENDIECK CATEGORIES AND THEIR DEFORMATIONS WITH AN APPLICATION TO SCHEMES

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Abstract. After presenting Grothendieck abelian categories as linear sites following [9], we present their basic deformation theory as developed in [14] and [10]. We apply the theory to certain categories of quasi-coherent modules over $\mathbb{Z}$-algebras, which can be considered as non-commutative projective schemes. The cohomological conditions we require constitute an improvement upon [5].

1. Introduction

This overview consists of two main sections. In §2, we introduce Grothendieck categories as the abelian categories closest to module categories. We explain how to extend the famous Gabriel-Popescu theorem in order to obtain other interesting representations of Grothendieck categories as linear sheaf categories. As an example, we give a sheaf theoretic description of categories of quasi-coherent modules, considered as non-commutative replacements of projective schemes in the approach to non-commutative algebraic geometry due to Artin, Tate, Stafford, Van den Bergh and others. In [3], $\mathbb{Z}$-algebras were used in order to describe non-commutative planes. In [22] and [23], Van den Bergh obtained explicit descriptions of the Grothendieck categories representing non-commutative planes, quadrics and $\mathbb{P}^1$-bundles over commutative schemes. The stability of these descriptions under deformation motivated the general development of a deformation theory for abelian, and in particular Grothendieck categories, which was started in [14]. In §3, we present the basics of this theory, and the application to quasi-coherent module categories. The details of this application can be found in [5]. In the current paper, we fine tune the approach in order to obtain applicability to all projective schemes with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

2. Grothendieck categories

2.1. **Linear categories.** In algebra, rings and algebras over fields or more general commutative ground rings are among the most basic objects of study. The algebraic geometry of affine schemes is entirely encoded in commutative ring theory. To model projective schemes, we will make use of an algebraic structure which is only slightly more general than that of an algebra. Throughout, let k be a commutative ground ring.

Definition 2.1. A *k-linear category* or *k-category* $\mathfrak{a}$ is a small category such that the hom-sets $\mathfrak{a}(A, B) = \mathrm{Hom}(A, B)$ for objects $A, B \in \mathfrak{a}$ are k-modules and the composition is k-bilinear.

Linear categories can be thought of as algebras with several objects, a point of view due to Mitchell [15].

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Examples 2.2.
(1) If A is a k-algebra, it can naturally be considered as a k-linear category with a single object $*$ and $\mathrm{End}(*, *) = A$.
(2) Let A and B be k-algebras and ${}_AM_B$ an A-B-bimodule. Then there is a corresponding k-linear category with two objects $*_A$ and $*_B$ and with $\mathrm{End}(*_A) = A$, $\mathrm{End}(*_B) = B$, $\mathrm{Hom}(*_A, *_B) = {}_AM_B$ and $\mathrm{Hom}(*_B, *_A) = 0$.
(3) The opposite category $\mathfrak{a}^{\mathrm{op}}$ of a k-linear category $\mathfrak{a}$ is again k-linear.

Remark 2.3. Linear categories offer alternative ways to organize algebraic structures that are classically organized into matrix algebras. For instance, Example 2.2 (2) corresponds to the matrix algebra

$$\begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}.$$

Let $\mathsf{Mod}(k)$ denote the category of k-modules.

Definition 2.4. A *k-linear functor* $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ between k-linear categories is a functor such that every $f_{A,A'} : \mathfrak{a}(A, A') \longrightarrow \mathfrak{b}(f(A), f(A'))$ is k-linear. A *right (resp. left) module over* $\mathfrak{a}$ or *right (resp. left)* $\mathfrak{a}$*-module* is a k-linear functor $\mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(k)$ (resp. $\mathfrak{a} \longrightarrow \mathsf{Mod}(k)$).

Remark 2.5. In this paper, we will always work with right modules and call them simply modules.

The category of $\mathfrak{a}$-modules is denoted by $\mathsf{Mod}(\mathfrak{a})$.

Example 2.6. If $\mathfrak{a}$ is the k-linear category associated to an algebra A as in Example 2.2 (1), then $\mathfrak{a}$-modules correspond precisely to right A-modules and $\mathsf{Mod}(\mathfrak{a}) \cong \mathsf{Mod}(A)$.

2.2. **Grothendieck categories.**

Grothendieck categories are the large abelian categories that are somehow closest to module categories. In this section we recall the definition and basic facts. For excellent introductions to the subject we refer the reader to the books [17] by Popescu and [21] by Stenström.

Definition 2.7. Let $\mathcal{C}$ be a cocomplete category and $\mathfrak{g}$ a set of objects in $\mathcal{C}$. We say that $\mathfrak{g}$ is a *set of generators for* $\mathcal{C}$ or that $\mathfrak{g}$ *generates* $\mathcal{C}$ or that $\mathfrak{g}$ *is (a) generating (set) (for* $\mathcal{C}$*)* if for every object $C \in \mathcal{C}$, there is an epimorphism $\oplus_{i \in I} G_i \longrightarrow C$ with I some index set and $G_i \in \mathfrak{g}$. A generating set consisting of a single object is simply called a *generator*.

Definition 2.8. A *Grothendieck category* is a cocomplete abelian category with exact directed colimits (i.e., directed colimits commute with finite limits) and a generating set.

Remark 2.9.
(1) In the original definition [8], Grothendieck used the "AB axioms" to define additional properties for abelian categories. In this terminology, cocompleteness is axiom AB3 and cocompleteness combined with exactness of directed colimits is axiom AB5.
(2) The fact that we require the collection of generators to be a *set* is crucial. Indeed, if we would allow classes of generators, then every category trivially has the collection of all its objects as a generating class.
(3) If a cocomplete category $\mathcal{C}$ has a set $\mathfrak{g}$ of generators, it also has a single generator G' obtained as

$$G' = \oplus_{G \in \mathfrak{g}} G.$$

Examples 2.10.
(1) The first and foremost examples of Grothendieck categories are module categories. Indeed, for a small linear category $\mathfrak{a}$, the representable functors

$$\mathfrak{a}(-, A) : \mathfrak{a} \longrightarrow \mathsf{Mod}(k) : B \longmapsto \mathfrak{a}(B, A)$$

for $A \in \mathfrak{a}$ consitute a generating set for $\mathsf{Mod}(\mathfrak{a})$, and, just like in ordinary module categories over rings, directed colimits are exact.
(2) For a ringed space $(X, \mathcal{O}_X)$, the category $\mathsf{Mod}(X)$ of scheaves of $\mathcal{O}_X$-modules on X is a Grothendieck category.
(3) For a quasi-compact, semi-separated scheme X, the category $\mathsf{Qch}(X)$ of quasi-coherent sheaves on X is Grothendieck.

The following theorem, due to Mitchel, characterizes module categories among Grothendieck categories:

Theorem 2.11. *Let $\mathcal{C}$ be a Grothendieck category and $\mathfrak{a} \subseteq \mathcal{C}$ a linear subcategory. The following are equivalent:*

(1) $\mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a}) : C \longmapsto \mathcal{C}(-, C)|_{\mathfrak{a}}$ *is an equivalence of categories.*
(2) $\mathfrak{a}$ *is a set of finitely generated projective generators of* $\mathcal{C}$*.*

2.3. **Gabriel-Popescu.** Let $\mathcal{C}$ be a Grothendieck category with a generator G, and put $A = \mathcal{C}(G, G)$, the endomorphism algebra of G in $\mathcal{C}$. By the famous Gabriel-Popescu theorem [18], $\mathcal{C}$ is a localization of $\mathsf{Mod}(A)$, more precisely the functor

$$\mathcal{C} \longrightarrow \mathsf{Mod}(A) : C \longmapsto \mathcal{C}(G, C)$$

is fully faithful and its left adjoint is exact. It follows that Grothendieck categories are precisely the localizations of module categories over algebras.

Of course, using different generators for $\mathcal{C}$, we can realize $\mathcal{C}$ as a localization of module categories over different rings. As we will see later on, it will be useful to also consider more general realizations of $\mathcal{C}$ as a localization of module categories over small linear categories. More precisely, we are interested in the following general setup:

Let $u : \mathfrak{a} \longrightarrow \mathcal{C}$ be a k-linear functor from a small k-category $\mathfrak{a}$ to a Grothendieck k-category $\mathcal{C}$. Consider the adjoint pair (a, i) with

$$i : \mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a}) : C \longmapsto \mathcal{C}(u(-), C)$$

and $a : \mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathcal{C}$ its left adjoint extending u over the Yoneda embedding $\mathfrak{a} \longrightarrow \mathsf{Mod}(\mathfrak{a})$.

We will say that the functor u is *localizing* provided (a, i) is a localization, i.e. i is fully faithful and a is exact.

2.4. **Linear topologies.** Let $\mathfrak{a}$ be a small k-category. Consider a representable $\mathfrak{a}(-, A) \in \mathsf{Mod}(\mathfrak{a})$ and a subobject $R \subseteq \mathfrak{a}(-, A)$ in $\mathsf{Mod}(\mathfrak{a})$. The subobject (also called *subfunctor*) corresponds to a *sieve* $\coprod_{A' \in \mathfrak{a}} R(A')$ of $\mathfrak{a}$-morphisms landing in A.

A *covering system* on $\mathfrak{a}$ consists of collections $\mathcal{T}(A)$ of subobjects of $\mathfrak{a}(-, A)$ in $\mathsf{Mod}(\mathfrak{a})$ for every $A \in \mathfrak{a}$. The subfunctors $R \in \mathcal{T}(A)$ are called *coverings* of A. The definition of a topology on $\mathfrak{a}$ is a linearized version (obtained by replacing Set by $\mathsf{Mod}(k)$ and enrichement over k of all involved notions, for instance replacement of the presheaf category $\mathsf{Fun}(\mathfrak{a}^{\mathrm{op}}, \mathsf{Set})$ by $\mathsf{Mod}(\mathfrak{a})$) of the notion of a Grothendieck topology on a small category [1] [4]. Consider the following conditions for a covering system $\mathcal{T}$ on $\mathfrak{a}$:

(1) $\mathcal{T}$ satisfies the *identity axiom* if $\mathfrak{a}(-, A) \in \mathcal{T}(A)$ for every $A \in \mathfrak{a}$.
(2) $\mathcal{T}$ satisfies the *pullback axiom* if for every $f : B \longrightarrow A$ in $\mathfrak{a}$ and $R \in \mathcal{T}(A)$, the pullback $f^{-1}R \subseteq \mathfrak{a}(-, B)$ is in $\mathcal{T}(B)$.
(3) $\mathcal{T}$ satisfies the *glueing axiom* if $S \in \mathcal{T}(A)$ as soon as there exists an $R \in \mathcal{T}(A)$ and for every $f : A_f \longrightarrow A$ in $R(A)$ an $R_f \in \mathcal{T}(A_f)$ with $R_f \subseteq f^{-1}S$.
(4) $\mathcal{T}$ is a *topology* if it satisfies the identity, pullback and glueing axioms.

With respect to a covering system $\mathcal{T}$ on $\mathfrak{a}$, a module $F : \mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(k)$ is called a *sheaf* provided every cover $R \subseteq \mathfrak{a}(-, A)$ induces a bijection

$$F(A) \cong \mathrm{Hom}(\mathfrak{a}(-, A), F) \longrightarrow \mathrm{Hom}(R, F).$$

Remark 2.12. For a covering $R \subseteq \mathfrak{a}(-, A)$, a morphism $R \longrightarrow F$ corresponds to the datum of elements $(x_f)_{f\in R}$ with $x_f \in F(A_f)$ for $f : A_f \longrightarrow A$ in R, such that for $g : A_{fg} \longrightarrow A_f$ we have $x_{fg} = F(g)(x_f)$. Thus, just like for an ordinary Grothendieck topology, morphisms $R \longrightarrow F$ correspond to compatible collections of elements in F, and the sheaf property expresses that a compatible collection has a unique glueing $x \in F(A)$ with $F(f)(x) = x_f$ for every $f \in R$.

Let $\mathcal{T}$ be a topology on $\mathfrak{a}$, and let $\mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ be the category of sheaves on $\mathfrak{a}$ with respect to $\mathcal{T}$. Then the inclusion $i : \mathsf{Sh}(\mathfrak{a}, \mathcal{T}) \subseteq \mathsf{Mod}(\mathfrak{a})$ is part of a localization, and the exact left adjoint is given by "sheafification" with respect to $\mathcal{T}$. In fact, if $\mathcal{T}$ is a covering system which satisfies the identity and pullback axiom, there is a smallest topology $\mathcal{T}'$ on $\mathfrak{a}$ containing $\mathcal{T}$, and for this topology $\mathsf{Sh}(\mathfrak{a}, \mathcal{T}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}')$. Imposing the glueing axiom rigidifies the situation in the following sense:

Proposition 2.13. *Let $\mathfrak{a}$ be a k-linear category. Taking sheaf categories defines a 1-1-correspondence between:*

(1) *Topologies on $\mathfrak{a}$;*
(2) *Localizations of* $\mathsf{Mod}(\mathfrak{a})$ *(up to equivalence of categories).*

For a given localization $i : \mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a})$ with left adjoint $a : \mathsf{Mod}(\mathfrak{a}) \longrightarrow \mathcal{C}$, the corresponding topology $\mathcal{T}_{\mathcal{C}}$ on $\mathfrak{a}$ consists of the $\mathcal{C}$*-epimorphic* subfunctors $R \subseteq \mathfrak{a}(-, A)$, i.e. the subfunctors R for which

$$\oplus_{f\in R(A_f)} u(A_f) \longrightarrow u(A)$$

is an epimorphism in $\mathcal{C}$. This defines an inverse to the map sending a topology $\mathcal{T}$ to the localization $\mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ of $\mathsf{Mod}(\mathfrak{a})$ (see [4] for details in a more general enriched setup).

Before stating the main theorem of this section, we introduce relative versions of some familiar notions.

Definition 2.14. Let $\mathcal{T}$ be a covering system on $\mathfrak{a}$ and let $f : M \longrightarrow N$ be a morphism in $\mathsf{Mod}(\mathfrak{a})$.

(1) f is a $\mathcal{T}$*-epimorphism* if the following holds: for every $y \in N(A)$ there is an $R \in \mathcal{T}(A)$ such that $N(g)(y) \in N(A_g)$ is in the image of $f_{A_g} : M(A_g) \longrightarrow N(A_g)$ for every $g : A_g \longrightarrow A$ in R.
(2) f is a $\mathcal{T}$*-monomorphism* if the following holds: for every $x \in M(A)$ with $f_A(x) = 0 \in N(A)$, there is an $R \in \mathcal{T}(A)$ such that $M(g)(x) = 0 \in M(A_g)$ for every $g : A_g \longrightarrow A$ in R.

Definition 2.15. Consider a linear functor $u : \mathfrak{a} \longrightarrow \mathcal{C}$ from a small k-linear category $\mathfrak{a}$ to a Grothendieck category $\mathcal{C}$ and a covering system $\mathcal{T}$ on $\mathfrak{a}$.

(1) u is *generating* if the images $u(A)$ for $A \in \mathfrak{a}$ are a collection of generators for $\mathcal{C}$.
(2) u is $\mathcal{T}$-full if for every $A \in \mathfrak{a}$ the canonical morphism $\mathfrak{a}(-, A) \longrightarrow \mathcal{C}(u(-), u(A))$ is a $\mathcal{T}$-epimorphism.
(3) u is $\mathcal{T}$-faithful if for every $A \in \mathfrak{a}$ the canonical morphism $\mathfrak{a}(-, A) \longrightarrow \mathcal{C}(u(-), u(A))$ is a $\mathcal{T}$-monomorphism.
(4) u is $\mathcal{T}$*-projective* if for every $\mathcal{C}$-epimorphism $c : X \longrightarrow Y$, the morphism
$$i(c) : \mathcal{C}(u(-), X) \longrightarrow \mathcal{C}(u(-), Y)$$
is a $\mathcal{T}$-epimorphism.

(5) u is *$\mathcal{T}$-finitely presented* if for every filtered colimit $\mathrm{colim}_i X_i$ in $\mathcal{C}$ the canonical morphism

$$\phi : \mathrm{colim}_i \mathcal{C}(u(-), X_i) \longrightarrow \mathcal{C}(u(-), \mathrm{colim}_i X_i)$$

is a $\mathcal{T}$-epimorphism and a $\mathcal{T}$-monomorphism.

(6) u is *$\mathcal{T}$-ample* if for every $R \in \mathcal{T}(A)$, the canonical morphism

$$\oplus_{f \in R(A_f)} u(A_f) \longrightarrow u(A)$$

is a $\mathcal{C}$-epimorphism.

Theorem 2.16. [5] *Consider $u : \mathfrak{a} \longrightarrow \mathcal{C}$ as above and let $\mathcal{T}$ be a topology on $\mathfrak{a}$. The following are equivalent:*

(1) *u induces a localization and $i : \mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a})$ factors through an equivalence $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$.*

(2) *u is generating, $\mathcal{T}$-full, $\mathcal{T}$-faithful, $\mathcal{T}$-projective, $\mathcal{T}$-finitely presented and $\mathcal{T}$-ample.*

Remarks 2.17. (1) Theorem 2.16 can be decomposed into two parts. First, the case where we take $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ (we know by Proposition 2.13 that this is actually the only possibility for $\mathcal{T}$ in (1)). In this case, we automatically get $\mathcal{T}_{\mathcal{C}}$-projectivity, $\mathcal{T}_{\mathcal{C}}$ -finitely presentedness and $\mathcal{T}_{\mathcal{C}}$-ampleness. The resulting characterization of localizing functors u was first obtained in [9]. Second, if the $\mathcal{T}$ we start from is arbitrary, these additional conditions are intended to ensure that $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$.

(2) If we take $\mathcal{T} = \mathcal{T}_{\mathrm{triv}}$ the trivial topology on $\mathfrak{a}$, for which the only coverings are the representable functors, Theorem 2.16 becomes Theorem 2.11.

2.5. **Quasi-coherent modules.** Next we look at some applications of §2.4 to schemes.

First, let us recall the situation for affine schemes. For a scheme $X = \mathsf{Spec}(A)$ with A a commutative ring, we have

$$\mathsf{Qch}(X) \cong \mathsf{Mod}(A).$$

Thus, the relevant Gothendieck categories are precisely module categories over commutative rings.

Next, we look into the situation for projective schemes. Consider a projective scheme $X = \mathrm{Proj}(A)$ for some positively graded algebra $A = (A_i)_{i \in \mathbb{Z}}$ with $A_i = 0$ for $i < 0$. By Serre's theorem [19], we have

$$\mathsf{Qch}(X) \cong \mathsf{Qgr}(A),$$

Where $\mathsf{Qgr}(A) = \mathsf{Gr}(A)/\mathsf{Tors}(A)$ is the quotient of the category $\mathsf{Gr}(A)$ of graded right A-modules by the category $\mathsf{Tors}(A)$ of torsion modules, i.e filtered colimits of right bounded modules.

Our aim is to describe $\mathsf{Qgr}(A)$ in terms of the tools of §2.4.

First, we look at $\mathsf{Gr}(\mathsf{A})$. Let A be a $\mathbb{Z}$-graded k-algebra and let $\mathsf{Gr}(A)$ be the category of $\mathbb{Z}$-graded right A-modules. Let (1) be the shift to the left on $\mathsf{Gr}(A)$, $(n) = (1)^n$, and consider the shifted objects $(A(n))_{n \in \mathbb{Z}}$ in $\mathsf{Gr}(A)$. For any $M \in \mathsf{Gr}(A)$, we have

$$\mathsf{Gr}(A)(A(n), M) \cong M_{-n}$$

and consequently the objects $A(n)$ constitute a set of finitely generated projective generators of $\mathsf{Gr}(A)$. Let $\mathfrak{a} = \mathfrak{a}(A)$ be the full linear subcategory of $\mathsf{Gr}(A)$ spanned by the $(A(n))_{n \in \mathbb{Z}}$, and rename the object $A(-n)$ by n. We then have

$$\mathfrak{a}(n, m) = \mathsf{Gr}(A)(A(-n), A(-m)) = A_{n-m}.$$

There is an induced equivalence of categories

$$\mathsf{Gr}(A) \cong \mathsf{Mod}(\mathfrak{a}) : M \longmapsto \mathsf{Gr}(A)(A(-?), M) = M_?$$

by Theorem 2.11.

A linear category with $\mathrm{Ob}(\mathfrak{a}) = \mathbb{Z}$ is called a *$\mathbb{Z}$-algebra* in [3]. If moreover $\mathfrak{a}(n, m) = 0$ unless $n \geq m$, then $\mathfrak{a}$ is called a *positively graded $\mathbb{Z}$-algebra*. Thus for a positively graded algebra A, the associated $\mathfrak{a}(A)$ is a positively graded $\mathbb{Z}$-algebra.

From now on, we let $\mathfrak{a}$ be an arbitrary positively graded $\mathbb{Z}$-algebra. We will now define a localization $\mathsf{Qmod}(\mathfrak{a})$ of $\mathsf{Mod}(\mathfrak{a})$ by means of a linear topology on $\mathfrak{a}$, which recovers $\mathsf{Qgr}(A)$ for $\mathfrak{a} = \mathfrak{a}(A)$.

For $m \in \mathbb{Z}$, consider the subobject

$$\mathfrak{a}(-, m)_{\geq n} \subseteq \mathfrak{a}(-, m)$$

defined by

$$\mathfrak{a}(k, m)_{\geq n} = \begin{cases} \mathfrak{a}(k, m) & \text{if } k \geq n \\ 0 & \text{otherwise.} \end{cases}$$

We first define the covering system $\mathcal{L}_{\mathrm{tails}}$ on $\mathfrak{a}$ for which $R \in \mathcal{L}_{\mathrm{tails}}(m)$ if and only if $\mathfrak{a}(-, m)_{\geq n} \subseteq R$ for some $m \leq n \in \mathbb{Z}$.

It is easy to see ([5, Proposition 3.9]) that $\mathcal{L}_{\mathrm{tails}}$ satisfies the identity and pullback axioms. We define the *tails topology* $\mathcal{T}_{\mathrm{tails}}$ on $\mathfrak{a}$ to be the smallest topology on $\mathfrak{a}$ containing $\mathcal{L}_{\mathrm{tails}}$. The *category of quasi-coherent modules over* $\mathfrak{a}$ is by definition

$$\mathsf{Qmod}(\mathfrak{a}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathrm{tails}}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{L}_{\mathrm{tails}}).$$

Remark 2.18. In general, $\mathcal{L}_{\mathrm{tails}}$ fails to be a topology (see [5, Example 3.12]), but in many cases of interest, it actually is a topology. These cases include the case where $\mathfrak{a}$ is noetherian and the case where $\mathfrak{a}$ is positively graded, connected (i.e. $\mathfrak{a}(n, n) = k$ for all n) and finitely generated in the sense of [5, §3.2]. In particular, this last case includes the $\mathfrak{a}(A)$ for positively graded, connected finitely generated graded algebras A, and moreover we then have $\mathsf{Qmod}(\mathfrak{a}(A)) \cong \mathsf{Qgr}(A)$.

2.6. **A characterization.** Let $\mathcal{C}$ be a Grothendieck category and let $(\mathcal{O}(n))_{n\in\mathbb{Z}}$ be a collection of objects in $\mathcal{C}$. Furthermore, let $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function with $\nu(n) \geq n$ for all $n \in \mathbb{Z}$.

We define a $\mathbb{Z}$-algebra $\mathfrak{a}$ with $\mathrm{Ob}(\mathfrak{a}) = \mathbb{Z}$ and

$$\mathfrak{a}(n, m) = \begin{cases} \mathcal{C}(\mathcal{O}(-n), \mathcal{O}(-m)) & \text{if } n \geq \nu(m) \\ 0 & \text{otherwise} \end{cases}$$

so that we obtain a natural functor

$$u : \mathfrak{a} \longrightarrow \mathcal{C} : n \longmapsto \mathcal{O}(-n).$$

The case where $\nu = 1_{\mathbb{Z}}$ is contained in [5]. The refinement of the results involving an arbitrary ν is almost for free, and will be important when we discuss deformations in §3.

Lemma 2.19. [5, Lemma 3.13] *The functor $u : \mathfrak{a} \longrightarrow \mathcal{C}$ is $\mathcal{T}_{\mathrm{tails}}$-full and $\mathcal{T}_{\mathrm{tails}}$-faithful.*

Proof. The functor u is faithful by construction, whence certainly $\mathcal{T}_{\mathrm{tails}}$-faithful. Consider the canonical maps

$$\varphi_{n,m} : \mathfrak{a}(n, m) \longrightarrow \mathcal{C}(\mathcal{O}(-n), \mathcal{O}(-m)).$$

For $n \geq \nu(m)$, $\varphi_{n,m}$ is an isomorphism by construction and nothing needs to be checked. So take $n < \nu(m)$ and consider a map $c : \mathcal{O}(-n) \longrightarrow \mathcal{O}(-m)$ in $\mathcal{C}$.

Consider the $\mathcal{T}_{\text{tails}}$-covering $\mathfrak{a}(-,n)_{\geq \nu(m)}$. For every $0 \neq x \in \mathfrak{a}(k,n)_{\geq \nu(m)}$, with consequently $k \geq \nu(m)$, we look at the composition

$$cu(x) : \mathcal{O}(-k) \longrightarrow \mathcal{O}(-m).$$

Since $k \geq \nu(m)$, we have $cu(x)$ in the image of $\varphi_{k,m}$, as desired. □

Theorem 2.20. [5, Theorem 3.15] *Let $\mathcal{C}$ be a Grothendieck category, $(\mathcal{O}(n))_{n\in\mathbb{Z}}$ a collection of objects in $\mathcal{C}$, and $u : \mathfrak{a} \longrightarrow \mathcal{C}$ as defined above. Suppose $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$ on $\mathfrak{a}$. The following are equivalent:*

(1) *The functor $u : \mathfrak{a} \longrightarrow \mathcal{C}$ induces an equivalence $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$.*
(2) *The following conditions are fulfilled:*
   (a) *the objects $\mathcal{O}(n)$ generate $\mathcal{C}$, i.e. for every $C \in \mathcal{C}$ there is an epimorphism*
   $$\oplus_i \mathcal{O}(n_i) \longrightarrow C.$$
   (b) *u is $\mathcal{L}_{\text{tails}}$-ample, i.e. for every $m \leq n$, there is an epimorphism*
   $$\oplus_i \mathcal{O}(-n_i) \longrightarrow \mathcal{O}(-m)$$
   with $n_i \geq n$ for every i.
   (c) *u is $\mathcal{L}_{\text{tails}}$-projective, i.e for every element $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(-m), M)$ with $m \in \mathbb{Z}$ and $M \in \mathcal{C}$, there is an $n_0 \geq m$ such that for every morphism $\mathcal{O}(-n) \longrightarrow \mathcal{O}(-m)$ with $n \geq n_0$, the natural image of ξ in $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(-n), M)$ is zero.*
   (d) *u is $\mathcal{L}_{\text{tails}}$-finitely presented, i.e. for every filtered colimit $\mathrm{colim}_i X_i$ in $\mathcal{C}$ and morphism $f : \mathcal{O}(-m) \longrightarrow \mathrm{colim}_i X_i$, there is an $n_0 \geq m$ such that for every $n \geq n_0$ every composition $\mathcal{O}(-n) \longrightarrow \mathcal{O}(-m) \longrightarrow \mathrm{colim}_i X_i$ factors through $\mathcal{O}(-n) \longrightarrow \mathcal{O}(-m) \longrightarrow X_i$ for some i. Moreover if a morphism $f : \mathcal{O}(-m) \longrightarrow X_i$ becomes zero when extended to $\mathrm{colim}_i X_i$, then there is an $n_0 \geq m$ such that for every $n \geq n_0$ every composition $\mathcal{O}(-n) \longrightarrow \mathcal{O}(-m) \longrightarrow X_i$ becomes zero when composed with a suitable $X_i \longrightarrow X_j$.*

Proof. This follows from Theorem 2.16 and Lemma 2.19. □

When we restrict the situation a bit, we recover the classical geometric notion of ampleness (condition (ab)):

Corollary 2.21. [5, Corollary 3.16] *Let $\mathcal{C}$ be a locally finitely presented Grothendieck category, $(\mathcal{O}(n))_{n\in\mathbb{Z}}$ a collection of finitely presented objects in $\mathcal{C}$, and $u : \mathfrak{a} \longrightarrow \mathcal{C}$ as defined above. Suppose $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$ on $\mathfrak{a}$. The following are equivalent:*

(1) *The functor $u : \mathfrak{a} \longrightarrow \mathcal{C}$ induces an equivalence $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$.*
(2) *The following conditions are fulfilled:*
   (ab) *$(\mathcal{O}(n))_{n\in\mathbb{Z}}$ is ample, i.e. for every finitely presented object $C \in \mathcal{C}$, there is an n_0 such that for every $n \geq n_0$, there is an epimorphism*
   $$\oplus_i \mathcal{O}(-n_i) \longrightarrow C$$
   with $n_i \geq n$ for every i.
   (c) *u is $\mathcal{L}_{\text{tails}}$-projective, i.e for every element $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(-m), M)$ with $m \in \mathbb{Z}$ and $M \in \mathcal{C}$, there is an $n_0 \geq m$ such that for every morphism $\mathcal{O}(-n) \longrightarrow \mathcal{O}(-m)$ with $n \geq n_0$, the natural image of ξ in $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(-n), M)$ is zero.*

To end this section, let us briefly return to the most classical geometric setup where X is a projective scheme over a noetherian base ring S, and see how Serre's original result fits in. The category $\mathcal{C} = \mathsf{Qch}(X)$ of quasi-coherent sheaves is locally finitely presented and has the category $\mathsf{coh}(X)$ of coherent sheaves as finitely presented objects.

Recall that an invertible sheaf $\mathcal{L}$ on X is called *ample* if for every coherent sheaf M, there is an n_0 such that for every $n \geq n_0$ there is an epimorphism

$$\oplus_i \mathcal{L}^{-n} \longrightarrow M.$$

Hence, putting $\mathcal{O}(n) = \mathcal{L}^n$, the collection $(\mathcal{O}(n))_{n\in\mathbb{Z}}$ satisfies condition (ab) in Corollary 2.21. Furthermore, by the cohomological criterion for ampleness, $\mathcal{L}$ is ample if and only if for every coherent sheaf M there is an n_0 such that for each $i > 0$ and for each $n \geq n_0$,

$$\mathrm{Ext}^i(\mathcal{L}^{-n}, M) = 0.$$

Thus, the collection also satisfies condition (c) and we recover Serre's original result.

Remark 2.22. Note that in the results of this section, as well as in the versions in [5], the main novelty is the sheaf theoretic approach to the proofs (by invoking Theorem 2.16). Indeed, Serre's original result was first generalized to the non-commutative setting using graded algebras by Artin an Zhang in [2], and later to $\mathbb{Z}$-algebras by Stafford and Van den Bergh [20] and Polishchuk [16].

3. Deformations of Grothendieck categories

In this section, we present the basic deformation theory of Grothendieck abelian categories as developed in [14] and [10] and discuss the application to categories of quasi-coherent modules and thus to projective schemes.

Our deformation setup is the following. Undeformed objects live over a commutative ground ring k, and we deform in the direction of artin local k-algebras R with maximal ideal m. Deformations in the direction of the dual numbers $k[\epsilon]$ with $\epsilon^2 = 0$ are called *first order deformations.*

3.1. **Algebras.**

Every non-commutative algebraic deformation theory is somehow based upon the deformation theory of algebras due to Gerstenhaber [6, 7]. The fundamental notions are the following.

Definition 3.1. Let A be a k-flat k-algebra. An *R-deformation* of A is an R-flat R-algebra B with an isomorphism $k \otimes_R B \cong A$ of k-algebras. An *equivalence* of R-deformations B and B' is an isomorphism $B \longrightarrow B'$ of R-algebras which reduces to the identity $1_A : A \longrightarrow A$ via the isomorphisms $k \otimes_R B \cong A$ and $k \otimes_R B' \cong A$.

Through deformation, commutative algebras can be turned into non-commutative algebras. For example, the commutative k-algebra $k[x, y]$ has non-commutative first order deformations given by

$$k\langle x, y\rangle/(xy - yx - f(x, y)\epsilon)$$

with $f(x, y) \in k[x, y]$.

From the geometric point of view, a commutative k-algebra corresponds to the affine scheme $\mathrm{Spec}(A)$, and this leads us to consider non-commutative R-deformations of A as "non-commutative affine R-schemes".

The deformation theory of a k-algebra A is controlled by its Hochschild complex. In particular, first order deformations of A are parameterized by the second Hochschild cohomology group $HH^2(A) = \mathrm{Ext}^2_{A-A}(A, A)$.

3.2. **Linear categories.** After having argued in §2.1 that linear categories can be considered as algebras with several objects, it is no surprise that a good deformation theory for these objects follows this philosophy. First, a k-linear category $\mathfrak{a}$ is k-flat provided all the Hom modules $\mathfrak{a}(A, A')$ for $A, A' \in \mathfrak{a}$ are k-flat. The reduction $k \otimes_R \mathfrak{b}$ of an R-linear category is the category with the same object set and reduced Hom modules.

Definition 3.2. Let $\mathfrak{a}$ be a k-flat k-linear category. An *R-deformation* of $\mathfrak{a}$ is an R-flat R-linear category $\mathfrak{b}$ with an isomorphism $k \otimes_R \mathfrak{b} \cong \mathfrak{a}$ of k-algebras. An *equivalence* of R-deformations $\mathfrak{b}$ and $\mathfrak{b}'$ is an isomorphism $\mathfrak{b} \longrightarrow \mathfrak{b}'$ of R-linear categories which reduces to the identity $1_{\mathfrak{a}} : \mathfrak{a} \longrightarrow \mathfrak{a}$ via the isomorphisms $k \otimes_R \mathfrak{b} \cong \mathfrak{a}$ and $k \otimes_R \mathfrak{b}' \cong \mathfrak{a}$.

Note that deformations of linear categories preserve the object set of the category.

Completely analogous to the algebra case, one can define a Hochschild complex for linear categories which controls their deformation theory (see [13] for the details).

3.3. **Abelian categories.** Although abelian categories are special cases of linear categories, the notion of linear deformation of §3.2 is not appropriate for abelian categories. To come up with a good notion for abelian categories, we first look at module categories over algebras. The main requirement for a deformation theory of abelian categories is the existence of a natural map

$$\mathrm{Def}_{\mathrm{alg}}(A) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(A)) : B \longmapsto \mathsf{Mod}(B)$$

from algebra deformations of A to abelian deformations of $\mathsf{Mod}(B)$. Clearly, if we compare $\mathsf{Mod}(A)$ to $\mathsf{Mod}(B)$, we observe that the object set is changed, and actually the relation can be described in the following way:

$$\mathsf{Mod}(A) \cong \{M \in \mathsf{Mod}(B) \mid mM = 0\}.$$

This brings us to the following natural definition. For an abelian R-category $\mathcal{B}$, we define the k-reduction to be the full (abelian!) subcategory

$$\mathcal{B}_k = \{B \in \mathcal{B} \mid mB = \mathrm{Im}(m \otimes_R B \longrightarrow B) = 0\}.$$

Furhermore, in [14, Definition 3.2], we introduce a notion of flatness for abelian categories which is such that a k-algebra A is k-flat if and only if its module category $\mathsf{Mod}(A)$ is abelian flat.

Definition 3.3. Let $\mathcal{A}$ be a flat abelian k-category. An *abelian R-deformation* of $\mathcal{A}$ is a flat abelian R-category $\mathcal{B}$ with an equivalence $\mathcal{A} \cong \mathcal{B}_k$. An *equivalence* of abelian R-deformations $\mathcal{B}$ and $\mathcal{B}'$ is an equivalence $\mathcal{B} \longrightarrow \mathcal{B}'$ of R-linear categories whose reduction is naturally isomorphic to the identity $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ via the equivalences $\mathcal{A} \cong \mathcal{B}_k$ and $\mathcal{A} \cong \mathcal{B}'_k$.

We have the following basic result:

Proposition 3.4. [14] *For a linear category $\mathfrak{a}$, there is a deformation equivalence*

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) : \mathfrak{b} \longrightarrow \mathsf{Mod}(\mathfrak{b})$$

from linear deformations of $\mathfrak{a}$ to abelian deformations of $\mathsf{Mod}(\mathfrak{a})$.

The main point in the proof is to associate a linear deformation of $\mathfrak{a}$ to a given abelian deformation $\mathcal{D}$ of $\mathcal{C} = \mathsf{Mod}(\mathfrak{a})$. Considering the objects $A \in \mathfrak{a}$ as objects of $\mathcal{C}$, we make essential use of the following two facts:

(1) $\mathrm{Ext}^1_{\mathcal{C}}(A, X \otimes_k A) = \mathrm{Ext}^2_{\mathcal{C}}(A, X \otimes_k A) = 0$ for all $A \in \mathfrak{a}$ and $X \in \mathsf{mod}(k)$ (in order to obtain unique flat lifts of the individual objects of $\mathfrak{a}$ along the left adjoint $k \otimes_R -$ of the embedding $\mathcal{C} \longrightarrow \mathcal{D}$);

(2) $\mathrm{Ext}^1_{\mathcal{C}}(A, X \otimes_k A') = 0$ for all $A, A' \in \mathfrak{a}$ and $X \in \mathsf{mod}(k)$ (in order to organize the lifted object as a linear deformation $\mathfrak{b} \subseteq \mathcal{D}$ of $\mathfrak{a}$).

Proposition 3.4 tells us that the non-commutative deformation theory of affine schemes is entirely controlled by Gerstenhaber's deformation theory for algebras.

For general abelian categories, an appropriate notion of Hochschild cohomology controling abelian deformations was introduced and studied in [13].

3.4. **Grothendieck categories.** In [14, Theorem 6.29], it was proven that abelian deformations of Grothendieck categories remain Grothendieck. In the proof, the axioms of a Grothendieck category are lifted one by one to a deformation, given that the original category is Grothendieck. The proof does not make use of representations of the original Grothendieck category as a localization of a module category. If we compare this result with Proposition 3.4 for module categories, clearly the latter contains a lot more information. For a given Grothendieck category, a first step in the good direction is to look for a set of generators and a localizing functor $u : \mathfrak{a} \longrightarrow \mathcal{C}$ such that there results a deformation equivalence between linear deformations of $\mathfrak{a}$ and abelian deformations of $\mathcal{C}$. In this respect we have the following key result from [14]:

Theorem 3.5. [14, Theorem 8.14] *Let $u : \mathfrak{a} \longrightarrow \mathcal{C}$ be a localizing functor from a small k-linear category $\mathfrak{a}$ to a Grothendieck k-category $\mathcal{C}$ such that the objects $u(A)$ are k-flat in $\mathcal{C}$. Suppose $\mathrm{Ob}(\mathfrak{a})$ is endowed with a transitive relation $\mathcal{R}$ such that*

(1) *For all $A \in \mathfrak{a}$ and $X \in \mathsf{mod}(k)$, we have*
$$\mathrm{Ext}^1_{\mathcal{C}}(u(A), X \otimes_k u(A)) = \mathrm{Ext}^2_{\mathcal{C}}(u(A), X \otimes_k u(A)) = 0;$$
(2) *$(A, A') \notin \mathcal{R}$ implies $\mathfrak{a}(A, A') = 0$;*
(3) *$(A, A') \in \mathcal{R}$ implies that $u_{(A,A')} : \mathfrak{a}(A, A') \longrightarrow \mathcal{C}(u(A), u(A'))$ is an isomorphism and that $\mathrm{Ext}^1_{\mathcal{C}}(u(A), X \otimes_k u(A')) = 0$.*

Then there is an equivalence of deformation functors $\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \cong \mathrm{Def}_{\mathrm{ab}}(\mathcal{C})$.

In fact, [14, Theorem 8.14] is more precise and describes both arrows constituting the deformation equivalence. This makes use of the fact that deformations can be "induced" upon localizations [14, §7].

3.5. **Quasi-coherent modules.** In this section, we apply Theorem 3.5 to the categories of quasi-coherent modules introduced in §2.5. We adopt the setup of §2.6.

Let $\mathcal{C}$ be a Grothendieck category and let $(\mathcal{O}(n))_{n \in \mathbb{Z}}$ be a collection of objects in $\mathcal{C}$. Let $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function with $\nu(n) \geq n$ for all $n \in \mathbb{Z}$. Define the $\mathbb{Z}$-algebra $\mathfrak{a}$ with $\mathrm{Ob}(\mathfrak{a}) = \mathbb{Z}$ and
$$\mathfrak{a}(n, m) = \begin{cases} \mathcal{C}(\mathcal{O}(-n), \mathcal{O}(-m)) & \text{if } n \geq \nu(m) \\ 0 & \text{otherwise} \end{cases}$$
and consider the natural functor
$$u : \mathfrak{a} \longrightarrow \mathcal{C} : n \longmapsto \mathcal{O}(-n).$$

Theorem 3.6. *Suppose the functor u induces an equivalence $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$, suppose the objects $\mathcal{O}(n)$ are flat and suppose for all n and $X \in \mathsf{mod}(k)$ we have*
$$\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(n), X \otimes_k \mathcal{O}(n)) = \mathrm{Ext}^2_{\mathcal{C}}(\mathcal{O}(n), X \otimes_k \mathcal{O}(n)) = 0$$
and for all $n \geq \nu(m)$ and $X \in \mathsf{mod}(k)$ we have
$$\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}(-n), X \otimes_k \mathcal{O}(-m)) = 0.$$
Then
$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C}) : \mathfrak{b} \longmapsto \mathsf{Qmod}(\mathfrak{b})$$

is an equivalence of deformation functors. More precisely, for every deformation $\mathcal{D}$ *of* $\mathcal{C}$ *there is a linear deformation* $\mathfrak{b}$ *of* $\mathfrak{a}$ *and a functor* $\mathfrak{b} \longrightarrow \mathcal{D}$ *satisfying the same conditions as* $\mathfrak{a} \longrightarrow \mathcal{C}$*.*

Proof. This is an application of Theorem 3.5. Clearly, the relation

$$(n,m) \in \mathcal{R} \iff n \geq \nu(m)$$

on $\mathrm{Ob}(\mathfrak{a})$ is transitive and satisfies the requirements (1) to (3) of the theorem by construction of u and by the assumptions. The given description of the deformation equivalence was proven in [5]. □

To end this section and overview, let us look at the geometric scope of the theorem. Let X be a projective scheme over a noetherian base ring with an ample invertible sheaf $\mathcal{L}$. Put $\mathcal{O}(n) = \mathcal{L}^n$. As discussed at the end of §2.6, the cohomological criterion of ampleness yields for every $m \in \mathbb{Z}$ a $\nu(m) \geq m$ such that for every $n \geq \nu(m)$, we have

$$\mathrm{Ext}^i(\mathcal{O}(-n), \mathcal{O}(-m)) = 0.$$

Thus, conditions (2) and (3) in the theorem hold for this choice of ν. Unfortunately, condition (1) - which is independent of ν - will not hold in general. It does hold under the additional condition that

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

Thus, for the class of projective schemes satisfying this restraint on their cohomology, all deformations can be described as "non-commutative projective schemes" over some deformed $\mathbb{Z}$-algebra.

Remarks 3.7.
(1) There exist other natural choices apart from taking $\mathcal{O}(n) = \mathcal{L}^n$. For instance for quadrics, a natural sequence of objects is given by

$$\dots, \mathcal{O}(n,n), \mathcal{O}(n+1,n), \mathcal{O}(n+1,n+1), \mathcal{O}(n+2,n+1), \dots$$

See [23] for a detailed geometric treatment of this case.
(2) The condition of the existence of a function ν making the necessary Ext's vanish naturally follows from a sort of "strong tails projectivity" condition. In [22], the combination of this condition and ampleness is called "stong ampleness".
(3) An approach to non-commutative deformations of schemes (and more general ringed spaces) based upon twisted deformations of the structure sheaf was developed in [11]. The relation between this approach and the one discussed in this paper, and a unified treatment of the two approaches based upon map-graded categories in the sense of [12], is work in progress.

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